IJCSS International Journal of Complex Systems in Science

Int. J. Complex Systems in Science
vol. **1(2)** (2011), pp. 169–173

Scaling Properties of the Explosive Percolation Transition

R. A. da Costa[1,†], S. N. Dogorovtsev[1,2], A. V. Goltsev[1,2] and J. F. F. Mendes[1]

[1] *Physics Department, University of Aveiro, 3810-193 Aveiro, Portugal*

[2] *A. F. Ioffe Physico-Technical Institute, 194021 St. Petersburg, Russia*

Abstract. The explosive percolation model was reported to undergo a discontinuous, first-order phase transition by Achlioptas et al. [1]. This original report was followed by numerous others, all claiming to observe the discontinuous explosive percolation transition. Subsequent studies of this class of irreversible process surprisingly revealed the presence of power-law critical cluster sizes distributions, as well as other features typical of continuous phase transition. This intriguing problem became an urgent issue in statistical physics. Our recent report [2] provided proof that the explosive percolation transition is continuous, thus resolving the apparent contradiction posed by the coexistence of a discontinuity with scaling behavior. Here we analyze a generalization of the dynamical rule of Ref. [2]. Besides proving the continuity of the percolation transition in all considered cases, we also describe in detail the scaling properties of this transition. We calculate critical exponents and derive relations between scaling function and critical exponents.

Keywords: Explosive percolation, Ordinary percolation, Phase transition.
MSC 2000: 60K35, 82C26, 82C43.

† **Corresponding author:** americo.costa@ua.pt
Received: October 14, 2011
Published: October 24, 2011

1. Introduction

Percolation transition is widely believed to be a typical continuous phase transition for various network architectures and space dimensionalities. So it shows standard scaling features, in a region close enough to the transition point, including power-law distribution of finite cluster sizes at the percolation threshold. One of the most studied percolation transitions occurs in the classical random graph, which is the best didactic example. In this model, hereafter referred to as ordinary percolation, at each time step an edge of the complete graph is occupied uniformly at random. Time t, in this problem, is the number of edges in the graph divided by the number of nodes N. This simple procedure produces a continuous percolation transition displaying scaling behavior, which can be exactly described when the system size $N \to \infty$. Close enough to the critical point, the size of the percolating giant cluster $S \propto (t - t_c)^\beta$, with the critical exponent $\beta = 1$ and the percolation threshold $t_c = 1/2$. The probability that a randomly chosen node belongs to a cluster of size $s \gg 1$ is $P(s, t) = s^{-\tau} f(s|t - t_c|^{1/\sigma})$, with $\tau = 3/2$ and $\sigma = 1/2$. The susceptibility is given by the average size of the cluster to which a randomly chosen node belongs: $\chi = \sum_s sP(s) \equiv \langle s \rangle_P \propto |t - t_c|^{-\gamma}$ with $\gamma = 1$.

Recently, however, it was reported that a remarkable percolation problem exists, in which the percolation cluster emerges discontinuously and already contains a finite fraction of nodes at the percolation threshold. To emphasize this surprising discontinuity this class of percolation systems was named explosive [1]. Further investigations of explosive percolation in the original, and other similar systems, based on numerical simulations, supported this strong result. But, in addition, surprisingly for discontinuous phase transitions, revealed the presence of many scaling behavior features, in particular, power-law critical distributions of cluster sizes [4, 5], consistent with continuous percolation transitions, see Fig. 1. The self-contradicting combination of discontinuity and scaling behavior has made explosive percolation one challenging and urgent issue in the physics of disordered systems.

This contradiction was resolved by considering a representative model which, unlike previous models, allows for an analytical treatment [2]. In this report we analyze a generalization of this model, which will be referred to as *explosive percolation model* throughout in this report. In addition, we extend our analysis to a model which mixes the rules of the explosive percolation and the ordinary percolation models, which will be referred to as *mix model* henceforth. For comparison we use the ordinary percolation model counterpart, see above.

2. Evolution Rules

According to the explosive percolation model at each time step we make the following. First we uniformly at random choose m nodes, then select that one of m which belongs to the smaller cluster. Afterwards we repeat this sampling procedure and select one more node, and finally add an edge connecting them.

In the mix model at each time step we add a new link between two nodes, each of the two selected by a different method. One node is selected uniformly at random, as in the ordinary percolation model. The other node is selected by choosing m nodes uniformly at random and taking that one of the m which belongs to the smaller cluster, as in the explosive percolation model. Note that for $m = 1$ this two models are equivalent to the ordinary percolation model.

3. Master Equations and Scaling

The master equation describing the evolution of the finite clusters size distribution of explosive percolation model is:

$$\frac{\partial P(s,t)}{\partial t} = \sum_{u+v=s} Q(u,t)Q(v,t) - 2sQ(s,t) \tag{1}$$

where $Q(s,t)$ is the probability that the size of the smallest of m clusters has size s at time t. Its definition in terms of the distribution $P(s,t)$ is:

$$Q(s,t) = \left(\sum_{u=s}^{\infty} P(u,t)\right)^m - \left(\sum_{u=s+1}^{\infty} P(u,t)\right)^m . \tag{2}$$

For the mix-model a similar equation can be written:

$$\frac{\partial P(s,t)}{\partial t} = \sum_{u+v=s} Q(u,t)P(v,t) - s\left[Q(s,t) + P(s,t)\right]. \tag{3}$$

It was possible to solve numerically the ordinary differential systems of Eqs. (1) and (3) for all s between 1 and s_{max}. The numerical calculation was performed for the ordinary percolation model ($m = 1$) and for the mix and explosive percolation models with $m = 2$. Fig. 1 shows $P(s)$ at different times t for these three cases. From this data we estimated the value of critical exponent β for the mix and explosive percolation models. When $m = 2$ we find that for the mix model $\beta = 0.2140(2)$. This value is between what was found for the explosive percolation model, where $\beta = 0.0555(1)$, and the value for the ordinary percolation model, where $\beta = 1/2$. The estimations of the critical point t_c and the set of critical exponents are shown in Table 1.

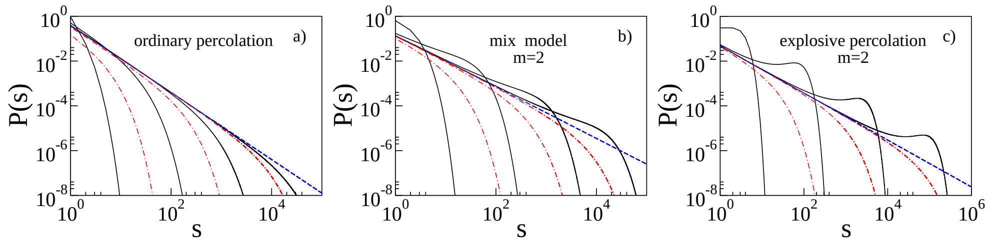

Figure 1: Finite cluster sizes distributions at different times t. The black solid curves correspond to $t < t_c$, the blue dashed curve to $t = t_c$, and the red dot-dashed curves correspond to $t > t_c$. For increasing time, before t_c, the tail of the distribution moves to the right. At t_c the distribution becomes power-law. After t_c it moves to the left. a) numerical solution of Eq. (1) with $m = 1$ and $s_{max} = 10^5$. b) numerical solution of Eq. (3) with $m = 2$ and $s_{max} = 10^5$, c) numerical solution of Eq. (1) with $m = 2$ and $s_{max} = 10^6$.

Table 1: Critical Points and Exponents for $m = 2$

Model	t_c	β	τ	σ	γ_P	γ_Q
ordinary	1/2	1	3/2	1/2	1	–
mix	0.797(1)	0.2140(2)	1.1499(1)	0.700(5)	1.214(1)	1
explosive	0.923(1)	0.0555(1)	1.0476(1)	0.857(3)	1.111(1)	1.055(1)

Writing generation functions for the distributions $P(s)$ and $Q(s)$ and following the same procedure as in [2, 3], for a power-law critical distribution $P(s, t_c) \propto s^{-\tau}$, one can show that the transition, in the mix and explosive percolation models, is continuous. The relative size of the giant cluster is $S(\delta) = B\delta^{\beta}$, where $\delta = |t - t_c|$. The coefficient B only depends on m and initial conditions. The exponent $\beta = (\tau - 1)/[1 - m(\tau - 1)]$ for the mix model and $\beta = (\tau - 1)/[1 - (2m - 1)(\tau - 1)]$ for the explosive percolation model.

In a second-order phase transition scaling behavior should emerge near the critical point, i.e. $P(s, \delta) = s^{-\tau} f(s\delta^{1/\sigma})$. Substituting this scaling form into Eq. (2) gives $Q(s, \delta) = s^{-1-m(\tau-1)} g(s\delta^{1/\sigma})$, where the scaling function $g(x) = mx^{(m-1)(\tau-1)} f(x) \left(\int_x^{\infty} dy y^{-\tau} f(y) + B\right)^{m-1}$. Introducing $\langle s\rangle_P \propto \delta^{-\gamma_P}$ and $\langle s\rangle_Q \propto \delta^{-\gamma_Q}$, and using the scaling forms of $P(s, \delta)$ and $Q(s, \delta)$ into Eqs. (1) and (3) one finds relations between critical exponents which allows us to write all exponents in terms of a single exponent, for both mix and explosive percolation models, see Table 2.

Table 2: Critical Exponents in Terms of β

Model	$\tau - 1$	σ	$\gamma_P - 1$	$\gamma_Q - 1$
mix	$\beta/(1+m\beta)$	$1/(1+m\beta)$	$(m-1)\beta$	0
explosive	$\beta/[1+(2m-1)\beta]$	$1/[1+(2m-1)\beta]$	$2(m-1)\beta$	$(m-1)\beta$

4. Conclusions

In the present paper we generalized the proof of continuity of the percolation transition of Ref. [2, 3] for arbitrary m. We also analyzed a model that mixes ordinary and explosive percolation models, and calculated the set of relations between critical exponents and scaling functions for mix and explosive percolation models. We found that the mix model undergoes a continuous phase transition, like the ordinary percolation and explosive percolation models, with intermediate exponents.

Acknowledgements

This work was partially supported by the projects PTDC/FIS/108476/2008, PTDC/SAU-NEU/103904/2008 and PTDC/MAT/114515/2009, and by FCT grand SFRH/BD/45628/2008.

References

[1] D. Achlioptas, R. M. D'Souza and J. Spencer, *Science* **323**, 1453 (2009).

[2] R. A. da Costa, S. N. Dogorovtsev, A. V. Goltsev and J. F. F. Mendes, *Phys. Rev. Lett.* **105**, 255701 (2010).

[3] R. A. da Costa, S. N. Dorogovtsev, A. V. Goltsev and J. F. F. Mendes, *arXiv*:1009.2534.

[4] F. Radicchi and S. Fortunato , *Phys. Rev. E* **81**, 036110 (2010).

[5] R. M. Ziff, *Phys. Rev. E* **82**, 051105 (2010).