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A method to approximate the hyperbolic sine of a matrix

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Abstract. In this paper a method for computing hyperbolic matrix sine based on Hermite matrix polynomial expansions is presented. An error bound analysis is given.

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1. Introduction

The hyperbolic sine of a matrix is defined by $\sinh(Ay) = \left(e^{Ay} - e^{-Ay}\right)/2$, $A \in \mathbb{C}^{r\times r}$. In general, functions of a matrix are used in many areas of science and engineering. In particular, matrix exponential e^A and matrix functions sine and cosine have been those that have received the most attention, not only for its computational difficulties but also for its importance in solving differential systems of first and second order, see for example [1, 2].
To approximate the hyperbolic sine of a matrix we need to compute approximations of the matrix exponential e^A and compute its inverse, e^{-A}, which may be costly numerical and computationally. A possible alternative is to use

$$\sinh(A) = i\cos\left(A - \frac{i\pi}{2}I\right),\ sinh(A) = -i\cosh\left(-A - \frac{i\pi}{2}I\right)$$

and approximate the cosine or the hyperbolic cosine of a matrix, but this approach has the disadvantage, however, to require complex arithmetic even though the matrix A is real – usually in applications– which contributes substantially to the computational overhead.

In this work we propose a method to evaluate $\sinh(A)$ avoiding complex arithmetic whenever possible. The proposed method use Hermite matrix polynomial expansions of $\sinh(A)$ in order to perform a very accurate and competitive method for computing them.
This work is organized as follows. Section 2 summarizes previous results of Hermite matrix polynomials and includes a new Hermite series expansion of the matrix hyperbolic sine with the respectively error bounds. An illustrative example is given in section 3.
Throughout this paper, $[x]$ denotes the integer part of x. The matrices I_r and $\theta_{r\times r}$ in $\mathbb{C}^{r\times r}$ denote the matrix identity and the null matrix of order r, respectively. For a matrix $C \in \mathbb{C}^{r\times r}$, we denote by $\|C\|$ its 2-norm defined in [3, p. 56] by

$$\|C\| = \sup_{x\neq 0} \frac{\|Cx\|_2}{\|x\|_2} \ , \tag{1}$$

where for a vector y in $\mathbb{C}^r$, $\|y\|_2$ denotes the usual euclidean norm.

2. Hermite matrix polynomial series expansions of hyperbolic matrix sine. Error bound.

For the sake of clarity in the presentation of the following results, we recall some properties of Hermite matrix polynomials which have been established in [4] and [5]. From (3.4) of [5] the $n-$th Hermite matrix polynomial is defined by

$$H_n\left(x, \frac{1}{2}A^2\right) = n! \sum_{k=0}^{[\frac{n}{2}]} \frac{(-1)^k (xA)^{n-2k}}{k!(n-2k)!} \ ,$$

for an arbitrary matrix A in $\mathbb{C}^{r\times r}$. Taking into account the three-term recurrence relationship (3.12) of [5, p. 26], it follows that

$$\left.\begin{array}{c} H_n\left(x, \frac{1}{2}A^2\right) = xAH_{n-1}\left(x, \frac{1}{2}A^2\right) - 2(n-1)H_{n-2}\left(x, \frac{1}{2}A^2\right) \quad , \quad n \geq 1 \\ H_{-1}(x, \frac{1}{2}A^2) = \theta_{r\times r} \quad , \quad H_0(x, \frac{1}{2}A^2) = I_r \end{array}\right] , \tag{2}$$

and from its generating function in (3.1) and (3.2) [5, p. 24] one gets

$$e^{xtA-t^2 I} = \sum_{n\geq 0} \frac{1}{n!} H_n\left(x, \frac{1}{2}A^2\right) t^n, \quad |t| < \infty, \tag{3}$$

where $x, t \in \mathbb{C}$. Then, after taking $y = tx$ and $\lambda = 1/t$ in (3) it follows that

$$e^{Ay} = e^{\frac{1}{\lambda^2}} \sum_{n \geq 0} \frac{1}{\lambda^n n!} H_n \left(\lambda y, \frac{1}{2} A^2 \right), \ \lambda \in \mathbb{C}, \quad y \in \mathbb{C}, \quad A \in \mathbb{C}^{r \times r} \ . \qquad (4)$$

Next, we wish to determine the series expansion in terms of Hermite matrix polynomials for the matrix hyperbolic sine $\sinh(Ay)$. Given an arbitrary matrix $A \in \mathbb{C}^{r\times r}$, using (4) in combination with [5, p. 25], it follows that $H_n(-x, A) = (-1)^n H_n(x, A)$. Thus, one obtains the following, required expression:

$$\sinh(Ay) \ = e^{\frac{1}{\lambda^2}} \sum_{n\geq 0} \frac{H_{2n+1}\left(y\lambda, \frac{1}{2}A^2\right)}{\lambda^{2n+1}(2n+1)!} \ . \qquad (5)$$

Denoting by $SH_N(A, \lambda)$ the Nth partial sum of series (5) for $y = 1$, one gets

$$SH_N(\lambda, A) = e^{\frac{1}{\lambda^2}} \sum_{n=0}^{N} \frac{H_{2n+1}\left(\lambda, \frac{1}{2}A^2\right)}{\lambda^{2n+1}(2n+1)!} \approx \sinh(A), \ \lambda \in \mathbb{C}, \quad A \in \mathbb{C}^{r\times r}. \qquad (6)$$

It is easy to show the following bound for Hermite matrix polynomials based on $||A^2|| \neq 0$, see [6], using the Taylor series for the hyperbolic sine:

$$\left\| H_{2n+1}\left(x, \frac{1}{2}A^2\right) \right\| \leq \frac{(2n+1)! e \, \|A\|}{\|A^2\|^{\frac{1}{2}}} \sinh\left(|x| \left\|A^2\right\|^{\frac{1}{2}} \right), \ \forall x \in \mathbb{R} \ , \ n \geq 0. \ (7)$$

Therefore, using (6) and bound (7), it follows that:

$$\left\| \sinh(A) - SH_N(\lambda, A^2) \right\| \leq e^{\frac{1}{\lambda^2}+1} \frac{\|A\|}{\|A^2\|^{\frac{1}{2}}} \frac{\sinh\left(|\lambda| \left\|A^2\right\|^{\frac{1}{2}} \right)}{(\lambda^2-1)\lambda^{2N+1}}. \qquad (8)$$

If $\lambda > 1$, given *a priori* error $\varepsilon > 0$ and taking the first positive integer N so that

$$N > \frac{\log\left(\frac{e^{\frac{1}{\lambda^2}+1} \|A\| \sinh\left(|\lambda| \left\|A^2\right\|^{\frac{1}{2}} \right)}{\varepsilon \|A^2\|^{\frac{1}{2}} (\lambda^2-1)} \right)}{2 \log(\lambda)} - 1 \qquad (9)$$

one gets

$$\left\| \sinh(A) - SH_N(\lambda, A^2) \right\| \leq \varepsilon.$$

3. An example.

Let A be a non-diagonalizable matrix defined by $A = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 1 & 2 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}$.
Using the minimal theorem [7, p. 571] the exact value of the hyperbolic matrix sine is

$$\sinh(A)$$
$$= \begin{bmatrix} 1.81343020392 & 0 & 0 & 1.81343020392 & 0 \\ 1.81343020392 & 0 & 0 & 1.81343020392 & 0 \\ 2.45165921420 & -1.17520119364 & 1.17520119364 & 3.62686040785 & 0 \\ 1.81343020392 & 0 & 0 & 1.81343020392 & 0 \\ 1.54680758967 & 0.80732175247 & 1.54308063482 & 1.91468703084 & 1.17520119364 \end{bmatrix}.$$

It is easy to check that $\|A\| \approx 3.42347$, $\left\|A^2\right\|^{\frac{1}{2}} \approx 2.83667$. Choosing $\lambda = 10$ and $\varepsilon = 10^{-5}$, the estimate (9) requires to take $N = 7$:

$$SH_7(10, A)$$
$$\approx \begin{bmatrix} 1.81343020383 & 0 & 0 & 1.81343020383 & 0 \\ 1.81343020383 & 0 & 0 & 1.81343020383 & 0 \\ 2.45165921402 & -1.17520119364 & 1.17520119364 & 3.62686040767 & 0 \\ 1.81343020383 & 0 & 0 & 1.81343020383 & 0 \\ 1.54680758940 & 0.80732175247 & 1.54308063482 & 1.91468703057 & 1.17520119364 \end{bmatrix},$$

with an error

$$\|\sinh(A) - SH_7(10, A)\| = 5.06825 \times 10^{-10}.$$

Note that the number of terms required to obtain a predetermined accuracy tends to be smaller than the one provided by (9). For instance, taking $N = 5$ and the same $\lambda = 10$ one finds

$$\|\sinh(A) - SH_5(10, A)\| = 2.52331 \times 10^{-6}.$$

The specific choice of parameter λ can still be further refined. Thus, for $N = 6$ and $\lambda = 6.21566$, one gets

$$\|\sinh(A) - SH_6(6.21566, A)\| = 1.33576 \times 10^{-8}.$$

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