

## On stability and controllability of multi-agent linear systems

M. I. García-Planas<sup>1,†</sup> and S. Tarragona<sup>2</sup>

<sup>1</sup> *Departament de Matemàtiques, Universitat Politècnica de Catalunya*

<sup>2</sup> *Departamento de Matemáticas, Universidad de León*

**Abstract.** Recent advances in communication and computing have made the control and coordination of dynamic network agents to become an area of multi-disciplinary research at the intersection of the theory of control systems, communication and linear algebra. The advances of the research in multi-agent systems are strongly supported by their critical applications in different areas as for example in consensus problem of communication networks, or formation control of mobile robots. Mainly, the consensus problem has been studied from the point of view of stability. Nevertheless, recently some researchers have started to analyze the controllability problems. The study of controllability is motivated by the fact that the architecture of communication network in engineering multi-agent systems is usually adjustable. Therefore, it is meaningful to analyze how to improve the controllability of a multi-agent system. In this work we analyze the stability and controllability of multiagent systems consisting of  $k + 1$  agents with dynamics  $\dot{x}^i = A_i x^i + B_i u^i$ ,  $i = 0, 1, \dots, k$ .

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† **Corresponding author:** maria.isabel.garcia@upc.edu

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### 1. Introduction

It is well known the great interest created in many research communities about the study of control multi-agents system, as well as the increasing interest in distributed control and coordination of networks consisting of multiple autonomous (potentially mobile) agents (see for example [3], [7], [9], [11]). It is due to the fact that the multi-agents appear in different areas as for example

in consensus problem of communication networks [8], or formation control of mobile robots [1].

Jinhuan Wang, Daizhan Cheng and Xiaoming Hu in [9] study the consensus problem in the case of multiagent systems in which all agents have an identical linear dynamics and it is a stable linear system. In [3], this result is generalized to the case where the dynamic of each agent is controllable.

Wei Ni and Daizhan Cheng in [6] analyze the case where  $A_0 = A_1 = \dots = A_k$  and  $B_1 = \dots = B_k$ ; this particular case has practical scenarios as the flight of groups of birds. It is obvious that in this case, the mechanic of the first system is independent of the others, then consensus under a fixed topology can be easily obtained and it follows from the motion of the first equation. This consensus problem is known as leader-following consensus problem [6], [4].

In this paper multiagent systems consisting of  $k$  agents of the same order with dynamics

$$\dot{x}^i = A_i x^i + B_i u^i \quad 1 \leq i \leq k \quad (1)$$

is considered.

## 2. Preliminaries

The topology of the system is defined by means of the well-known topology induced by a graph in the following manner: we consider a graph  $\mathcal{G} = (\mathcal{V}, \mathcal{E})$  of order  $k$  with the set of vertices  $\mathcal{V} = \{1, \dots, k\}$  and the set of edges  $\mathcal{E} = \{(i, j) \mid i, j \in \mathcal{V}\} \subset \mathcal{V} \times \mathcal{V}$ .

Given an edge  $(i, j)$   $i$  is called the parent node and  $j$  is called the child node and  $j$  is in the neighbor of  $i$ , specifically we define the neighbor of  $i$  and we denote it by  $\mathcal{N}_i$  to the set  $\mathcal{N}_i = \{j \in \mathcal{V} \mid (i, j) \in \mathcal{E}\}$ .

The graph is called undirected if verifies that  $(i, j) \in \mathcal{E}$  if and only if  $(j, i) \in \mathcal{E}$ . The graph is called connected if there exists a path between any two vertices, otherwise is called disconnected.

Associated to the graph we consider a matrix  $G = (g_{ij})$  called (unweighted) adjacency matrix defined as follows  $g_{ii} = 0$ ,  $g_{ij} = 1$  if  $(i, j) \in \mathcal{E}$ , and  $g_{ij} = 0$  otherwise.

In a more general case we can consider that a weighted adjacency matrix is  $G = (g_{ij})$  with  $g_{ii} = 0$ ,  $g_{ij} > 0$  if  $(i, j) \in \mathcal{E}$ , and  $g_{ij} = 0$  otherwise.

The Laplacian matrix of the graph is

$$\mathcal{L} = (l_{ij}) = \begin{cases} |\mathcal{N}_i| & \text{if } i = j \\ -1 & \text{if } j \in \mathcal{N}_i \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

Taking into account that the graph is undirected the matrix  $\mathcal{L}$  is symmetric, consequently there exists an orthogonal matrix  $P$  such that  $P\mathcal{L}P^t = \mathcal{D}$ . Moreover 0 is an eigenvalue of  $\mathcal{L}$  and  $(1, \dots, 1)^t$  is the associated eigenvector, and in the case where the graph is also connected the eigenvalue 0 is simple. (For more details about graph theory see [10]).

It is helpful to remember that given  $A = (a_{ij}) \in M_{n \times m}(\mathbb{C})$  and  $B = (b_{ij}) \in M_{p \times q}(\mathbb{C})$  the Kronecker product of the matrices is defined as

$$A \otimes B = (a_{ij}B) \in M_{np \times mq}(\mathbb{C}).$$

See [5] for more information and properties.

Finally, recall that the dynamical system  $\dot{x} = Ax + Bu$  is said to be controllable if for every initial condition  $x(0)$  and every vector  $x_1 \in \mathbb{R}^n$ , there exist a finite time  $t_1$  and control  $u(t) \in \mathbb{R}^m$ ,  $t \in [0, t_1]$ , such that  $x(t_1) = x_1$ .

This definition requires only that any initial state  $x(0)$  can be steered to any final state  $x_1$  at some time  $t_1$ . However, the trajectory of the dynamical system between 0 and  $t_1$  is not specified. Furthermore, there is no constraints posed on the control vector  $u(t)$  and the state vector  $x(t)$ .

It is well-known that the controllability can be ensured computing the rank of the matrix

$$\mathcal{C} = \begin{pmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{pmatrix}. \quad (3)$$

called controllability matrix, according to the following proposition.

**Proposition 1** *The dynamical system  $\dot{x} = Ax + Bu$  is controllable if and only if  $\text{rank } \mathcal{C} = n$ .*

As it has been said, controllability of the dynamical system  $\dot{x} = Ax + Bu$  implies that each initial state can be steered to 0 on a finite time-interval. If one only requires this condition to happen asymptotically for  $t \rightarrow \infty$ , one has the definition of stabilizable systems. That is to say:

The system  $\dot{x} = Ax + Bu$  is called stabilizable if for each initial state  $x_0 \in \mathbb{R}^n$  there exists a (piece-wise continuous) control input  $u : [0, \infty) \rightarrow \mathbb{R}^m$  such that the state-response with  $x(0) = x_0$  verifies

$$\lim_{t \rightarrow \infty} x(t) = 0.$$

(Stability for singular dynamical linear systems was studied by M.I. García-Planas in [2]).

### 3. Consensus problem

Roughly speaking, we can define the consensus as a collection of processes such that each process starts with an initial value, where each one is supposed to output the same value and there is a validity condition that relates outputs to inputs. More specifically, the consensus problem is a canonical problem that appears in the coordination of multi-agent systems.

Given initial values (scalar or vector) of agents, the objective is to establish conditions under which and through local interactions and computations, the agents asymptotically agree upon a common value; that is to say: to reach a consensus.

Let us consider now, a multi-agent where the dynamic of each agent is given by the following dynamical system as (1) where matrices  $A_i$  and  $B_i$  are not necessarily equal.

Communication topology among agents are defined in § 2.

The consensus of the system (1) is achieved using local information if there exists a state feedback

$$u^i = K_i \sum_{j \in \mathcal{N}_i} (x^i - x^j), \quad 1 \leq i \leq k$$

such that

$$\lim_{t \rightarrow \infty} \|x^i - x^j\| = 0, \quad 1 \leq i, j \leq k.$$

For simplicity we define  $z^i = \sum_{j \in \mathcal{N}_i} (x^i - x^j)$ ,  $1 \leq i \leq k$ .

The closed-loop system obtained under this feedback is as follows

$$\dot{\mathcal{X}} = \mathcal{A}\mathcal{X} + \mathcal{B}\mathcal{K}\mathcal{Z}$$

where

$$\mathcal{X} = (x^1 \quad \dots \quad x^k)^t, \quad \dot{\mathcal{X}} = (\dot{x}^1 \quad \dots \quad \dot{x}^k)^t$$

$$\mathcal{A} = \text{diag}(A_1, \dots, A_k), \quad \mathcal{B} = \text{diag}(B_1, \dots, B_k)$$

$$\mathcal{K} = \text{diag}(K_1, \dots, K_k), \quad \mathcal{Z} = \left( \sum_{j \in \mathcal{N}_1} x^1 - x^j \quad \dots \quad \sum_{j \in \mathcal{N}_k} x^k - x^j \right)^t.$$

From

$$\mathcal{Z} = (\mathcal{L} \otimes I_n)\mathcal{X}$$

the following proposition is deduced.

**Proposition 2 ([3])** *The closed-loop system can be deduced in terms of matrices  $\mathcal{A}$ ,  $\mathcal{B}$  and  $\mathcal{K}$  in the following manner.*

$$\dot{\mathcal{X}} = (\mathcal{A} + \mathcal{B}\mathcal{K}(\mathcal{L} \otimes I_n))\mathcal{X} \quad (4)$$

The interest is in  $K_i$  such that the consensus is achieved.

**Proposition 3 ([3])** *Let us consider the system (1) with a connected adjacent topology. If the system (1) is stable, the consensus problem has a solution.*

**Corollary 3** *Let us consider the system (1) with a connected adjacent topology. If the matrices  $A_i$  defining the dynamic are Hurwitz stable, then the consensus is achieved.*

**Remark 1** The system 4 can be written as

$$\dot{\mathcal{X}} = \mathcal{A}\mathcal{X} + \mathcal{B}\mathcal{U} \quad \text{with} \quad \mathcal{U} = \mathcal{K}(\mathcal{L} \otimes I_n)\mathcal{X}.$$

So, we have the following proposition.

**Proposition 4** *A necessary (but not sufficient) condition for consensus to be reached is that the system*

$$\dot{\mathcal{X}} = \mathcal{A}\mathcal{X} + \mathcal{B}\mathcal{U}$$

*is stabilizable.*

**Corollary 4**

A necessary condition for consensus to be reached is that the systems

$$\dot{x}^i = A_i x^i + B_i u^i, \quad \forall i = 1, \dots, k$$

are stabilizable.

**Remark 2** The feedback  $\mathcal{K}$  obtained from the feedbacks stabilizing the systems  $\dot{x}^i = A_i x^i + B_i u^i$  does not necessarily stabilize the system  $\dot{\mathcal{X}} = (\mathcal{A} + \overline{\mathcal{BK}}(\mathcal{L} \otimes I_n))\mathcal{X}$ .

**Example.**

Let us consider the following two one-dimensional systems

$$\begin{aligned} \dot{x}^1 &= u^1 \\ \dot{x}^2 &= x^2 + u^2 \end{aligned}$$

The communication topology is defined by the graph  $\mathcal{V} = \{1, 2\}$ ,  $\mathcal{E} = \{(1, 2)\} \subset \mathcal{V} \times \mathcal{V}$ .

Taking  $\mathcal{K} = \begin{pmatrix} 1 & -1 \\ 6 & -6 \end{pmatrix}$  we have  $(\mathcal{A} + \mathcal{BK}(\mathcal{L} \otimes I_n)) = \begin{pmatrix} 1 & -1 \\ 6 & -5 \end{pmatrix}$  with eigenvalues  $-0.2679$ , and  $-3.7321$ , then the system is stable.

Taking  $k_1 = -1$  and  $k_2 = -2$ , clearly these feedbacks stabilize the systems, but taking  $\mathcal{K} = \begin{pmatrix} k_1 & \\ & k_2 \end{pmatrix} = \begin{pmatrix} -1 & \\ & -2 \end{pmatrix}$  we have  $(\mathcal{A} + \mathcal{BK}(\mathcal{L} \otimes I_n)) = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}$  with eigenvalues  $0.4142$ , and  $-2.4142$ , then the system is not stable.

#### 4. Conclusions

In this work, it is showed that the stability and controllability of multiagent systems consisting of  $k + 1$  agents with dynamics  $\dot{x}^i = A_i x^i + B_i u^i$ ,  $i = 0, 1, \dots, k$  permit to obtain the consensus of the multisystem.

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