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# The marked type of an invariant subspace

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**Abstract.** We consider the study of $f$-invariant subspaces $\mathcal{V} \in \mathbb{C}^N$ for a fixed endomorphism $f \in End(\mathbb{C}^N)$. Our goal is introducing a new classification item for each invariant subspace, which we call its "marked type": in some sense the nearest marked subspace.

We start from the fact that any invariant subspace is a perturbation of some marked subspace, not unique in general. The key point is a criterium to select one of them, which we call its marked type. Thus, the study of $f$-invariant subspaces can be reduced to those having the same marked type.

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## 1. Introduction

Given a fixed endomorphism $f \in End(\mathbb{C}^N)$, a subspace $\mathcal{V} \subset \mathbb{C}^N$ is called $f$-*invariant* if $f(\mathcal{V}) \subset \mathcal{V}$, and two of them $\mathcal{V}$ and $\mathcal{V}'$ are called *equivalent* if there is an isomorphism $\varphi$ such that $\varphi(\mathcal{V}) = \mathcal{V}'$ and $\varphi \circ f = f \circ \varphi$. Without loss of generality it is clear that we can assume $f$ nilpotent. The classification of $f$-invariant subspaces according to this equivalence relation is an open problem, with partial results for quite particular cases: [10] and [4] for monogenic subspaces; in [11] one proves that there is a finite number of equivalence classes if the degree of the minimal polynomial is less or equal than 5, and one asserts that the classification problem is "wild" when this degree is greater than 6.

If $\mathcal{V}$ and $\mathcal{V}'$ are equivalent, then the restriction of $f$ to them must have the same Jordan form. Therefore we can fix the Segre characteristic $p = (p_1, \ldots, p_n)$ of $f$ and $q = (q_1, \ldots, q_n)$ of its restriction to the invariant subspace. We write $Inv(p, q)$ the set of these invariant subspaces (in [12] one shows that it is a differentiable manifold). Moreover, if $\mathcal{V}$ and $\mathcal{V}'$ are equivalent then also the quotient endomorphisms must have the same Segre characteristic, but there are not explicit criteria in order to determine if a quotient Segre characteristic is compatible with $p$ and $q$ (the Carlson problem: see [2], [3]).

Our goal is assigning to each element of $Inv(p,q)$ a marked subspace (we recall that a marked subspace is an invariant subspace having a Jordan basis of the restriction which can be extended to a Jordan basis of $f$ (see [9], [8]) which we call its *marked type*. From a geometric point of view the marked type of a non-marked class is the greatest marked class which intersects its boundary.

Hence, the set $Inv(p,q)$ is partitioned in a finite number of subsets, each one associated to a marked subspace. Therefore the study of invariant subspaces (in particular the Carlson problem, the classification problem, ...) can be reduced to those in $Inv(p,q)$ having the same marked type.

In addition, because of the fact that any invariant subspace is the perturbation of some marked one, by means of the Arnold's theory (see [1]) it is possible to obtain (see [7] and [6]) parameterized families, centered on each marked subspace, which contain representatives of the invariant subspaces having it as marked type (and only of them).

In this paper we present an example where the above ideas are reflected. The formal definitions and proofs will be detailed in future works.

## 2. The previous steps

Let us consider $f \in End(\mathbb{C}^N)$, nilpotent, having Segre characteristic $p = (3,2,1)$, and the invariant subspaces $\mathcal{V} \in Inv(p,q)$ such that the restriction of $f$ to $\mathcal{V}$ has the Segre characteristic $q = (2,1)$.

It is clear that the matrix of $f$ in any basis of $\mathbb{C}^6$ which extends a Jordan basis of $\mathcal{V}$ has the form

$$\left[\begin{array}{cc|c||ccc}
0 & & & * & * & * \\
1 & 0 & & * & * & * \\
\hline
 & & 0 & * & * & * \\
\hline
 & & & * & * & * \\
 & & & * & * & * \\
 & & & * & * & *
\end{array}\right]$$

Our goal is to classify them according to the values of the entries *.

Due to Arnold's theory, we can simplify this general matrix to a miniversal deformation of the marked central one (see [6]):

$$\left[\begin{array}{cc|c||ccc} 0 & & & * & * & * \\ 1 & 0 & & 0 & 0 & 0 \\ \hline & & 0 & * & * & * \\ \hline & & & * & * & * \\ & & & * & * & * \\ & & & * & * & * \end{array}\right]$$

However it is not easy to characterize, for example, the conditions which give the matrix being nilpotent with $p = (3, 2, 1)$.

An alternative approach (see [12]) is fixing a Jordan basis of $f$ and considering the matrix of change of basis to the ones above:

$$S = \left[\begin{array}{cc|c||ccc} & & & 1 & & \\ 1 & & & & & \\ x^1_{11} & 1 & x^1_{12} & & & \\ \hline x^2_{21} & & & & 1 & \\ x^1_{21} & x^2_{21} & 1 & & & \\ \hline x^1_{31} & & x^1_{32} & & & 1 \end{array}\right]$$

Then, if $J_p$ is the matrix of $f$ in any Jordan basis:

$$S^{-1} J_p S = \left[\begin{array}{cc|c||ccc} 0 & & & 1 & 0 & 0 \\ 1 & 0 & & * & * & 0 \\ \hline & & 0 & * & 1 & 0 \\ \hline & & & 0 & 0 & 0 \\ & & & * & 0 & 0 \\ & & & * & * & 0 \end{array}\right]$$

Again we can reduce $S$ to a miniversal deformation of the central marked subspace (see [5]):

$$\bar{S} = \left[\begin{array}{cc|c||ccc} & & & 1 & & \\ 1 & & & & & \\ & 1 & & & & \\ \hline x & & & & 1 & \\ & x & 1 & & & \\ \hline y & & z & & & 1 \end{array}\right], \qquad \bar{S}^{-1} J_p \bar{S} = \left[\begin{array}{cc|c||ccc} 0 & & & 1 & 0 & 0 \\ 1 & 0 & & 0 & 0 & 0 \\ \hline & & 0 & 0 & 1 & 0 \\ \hline & & & 0 & 0 & 0 \\ & & & -x & 0 & 0 \\ & & & -y & -z & 0 \end{array}\right]$$

However it is not trivial, in general, to discuss, for example, the Segre characteristic of the quotient endomorphism.

## 3. The novelties

Now let us consider the marked subspaces in $Inv((3,2,1),(2,1))$. Clearly they can be determined by reordering $p$: (3,2,1), (3,1,2), (2,3,1), (2,1,3), where the first index is the length of the Jordan chain that contains the Jordan chain of the subspace with length $q_1 = 2$ and the second corresponds to $q_2 = 1$.

They correspond, respectively, to:

(1) $x = y = z = 0$

(2) $z \neq 0, x = y = 0$

(3) $x \neq 0, y = z = 0$

(4) $x \neq 0, z \neq 0, y = 0$

For the first one, the miniversal change of basis and the corresponding matrix are:

$$S_1 = \left[\begin{array}{cc|c||ccc} & & & 1 & & \\ 1 & & & & & \\ & 1 & & & & \\ \hline & & & & 1 & \\ & & 1 & & & \\ \hline y & & & & & 1 \end{array}\right], \qquad S_1^{-1} J_p S_1 = \left[\begin{array}{cc|c||ccc} 0 & & & 1 & 0 & 0 \\ 1 & 0 & & 0 & 0 & 0 \\ \hline & & 0 & 0 & 1 & 0 \\ \hline & & & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & -y & 0 & 0 \end{array}\right]$$

All of them have marked type (1). Indeed, only one additional possibility appears:

$$(1') \quad x = z = 0, \quad y \neq 0$$

For the remainder cases, the miniversal changes of basis are constant, so that there are not equivalence classes having marked type (2), (3) and (4) (other than themselves).

Summarizing, in $Inv((3,2,1),(2,1))$ one has 5 equivalence classes:

$$A_1 = \left[\begin{array}{cc|c||ccc} & & & 1 & & \\ 1 & & & & & \\ \hline & & & & 1 & \\ \hline\hline & & & & & \\ & & & & & \\ & & & & & \end{array}\right], \quad A_2 = \left[\begin{array}{cc|c||c|cc} & & & 1 & & \\ 1 & & & & & \\ \hline & & & & & \\ \hline\hline & & & & & \\ \hline & & & & & \\ & & & & 1 & \end{array}\right], \quad A_3 = \left[\begin{array}{cc|c||cc|c} & & & & & \\ 1 & & & & & \\ \hline & & & & 1 & \\ \hline\hline & & & & & \\ & & & 1 & & \\ \hline & & & & & \end{array}\right]$$

$$A_4 = \left[\begin{array}{c|c||cc} & & & \\ 1 & & & \\ \hline & & & \\ \hline\hline & & 1 & \\ & & & 1 \end{array}\right], \qquad A_1' = \left[\begin{array}{c|c||c|c} & & 1 & \\ 1 & & & \\ \hline & & & 1 \\ \hline\hline & & 1 & \\ \hline & & & \end{array}\right]$$

The marked ones are $A_1, A_2, A_3, A_4$. The only non-marked class is $A_1'$, which has marked type $A_1$.

Notice (see [5]) that the codimensions of the marked classes are respectively 3,1,1,0. Only the first one intersects the boundary of the non-marked class.

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